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2009 J. Phys. A: Math. Theor. 42 155302

(http://iopscience.iop.org/1751-8121/42/15/155302)

IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 155302 (12pp) doi:10.1088/1751-8113/42/15/155302

A generalized Wigner function on the space of irreducible representations of the Weyl–Heisenberg group and its transformation properties

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Received 11 December 2008, in final form 28 January 2009
Published 20 March 2009
Online at stacks.iop.org/JPhysA/42/155302

Abstract
A natural extension of the Wigner function to the space of irreducible unitary representations of the Weyl–Heisenberg group is discussed. The action of the automorphisms group of the Weyl–Heisenberg group onto Wigner functions and their generalizations and onto symplectic tomograms is elucidated. Some examples of physical systems are considered to illustrate some aspects of the characterization of the Wigner functions as solutions of differential equations.

PACS numbers: 03.65−w, 03.65.Fd, 02.30.Uu

1. Introduction

The Wigner function [1] can be defined as the Weyl symbol [2] of a density state [3]. The Weyl symbol of any operator is defined in terms of the parity operator $\mathcal{P}$ and displacement operator $D(z)$ resulting from the construction of a Weyl system. Both operators are exponential functions of boson annihilation and creation operators, a and $a^\dagger$. The parity operator depends quadratically on them, $\mathcal{P} = \exp(\mathrm{i}\pi a^\dagger a)$, and the displacement linearly, $D(z) = \exp(za^\dagger - z^* a)$. The parameter z is a complex number which can be expressed in terms of phase space coordinates as $q = \sqrt{2}\,\mathrm{Re}\, z$, $p = \sqrt{2}\,\mathrm{Im}\, z$. Creation and annihilation operators are associated with a realization of the generators of the $(2+1)$-dimensional Weyl–Heisenberg group and $D(z)$ with its infinite-dimensional unitary representations. Generalization to the $(2n+1)$-dimensional case is straightforward.

There are various ways to generalize the construction of the Wigner function. One method consists of extending the parity operator by changing the angle appearing in its definition from

π to an arbitrary value $\theta : \mathcal{P}_\theta = \exp(i\theta a^\dagger a)$. In fact, this generalization is closely related to the construction of the so-called s-ordered quasi-distributions or Wigner functions [4].

Another generalization consists of considering the automorphism group of the Weyl–Heisenberg group and to extending the displacement operator $D(z)$. Because the displacement operators $D(z)$ are obtained from the irreducible unitary representations of the Weyl–Heisenberg group, we will have to study the action of the automorphism group on the space of their irreducible unitary representations, which are labeled by a real number γ. One of the aims of this work is to focus on this generalization.

The Radon transform [5] of the Wigner function was used to suggest a probability distribution description of quantum states [6]. These probability distributions, called symplectic tomograms of the quantum states, are related to Wigner functions by a linear integral transform. Then we can determine the action on symplectic tomograms of any transformation on the Wigner function. Thus another goal of this work will be to get the transformation properties of the symplectic tomograms induced by transformation of the Wigner function under the action of the group of the automorphisms of the Weyl–Heisenberg group.

2. Wigner functions and Weyl symbols of operators

Further insight on the physical meaning of the Wigner function of a density state ρ, defined as

$$\mathcal{W}_\rho(\mathbf{p}, \mathbf{q}) = \int_{\mathbb{R}^n} d^n x \, e^{-i\mathbf{p}\cdot\mathbf{x}} \langle \mathbf{q} + \mathbf{x}/2 | \rho | \mathbf{q} - \mathbf{x}/2 \rangle \tag{1}$$

(hereafter $\hbar = 1$ and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$) was obtained from its representation as the expectation value of the shifted parity operator $\mathcal{P}$ (see Royer [7]):

$$\mathcal{W}_\rho(\mathbf{p}, \mathbf{q}) = 2^n \operatorname{Tr}[\rho D(\mathbf{p}, \mathbf{q}) \mathcal{P} D^\dagger(\mathbf{p}, \mathbf{q})] = 2^n \operatorname{Tr}[\rho D(2\mathbf{p}, 2\mathbf{q}) \mathcal{P}]. \tag{2}$$

The displacement operators $D(\mathbf{p}, \mathbf{q})$ have the usual form

$$D(\mathbf{p}, \mathbf{q}) = \exp[i(\mathbf{p} \cdot \hat{\mathbf{Q}} - \mathbf{q} \cdot \hat{\mathbf{P}})], \qquad [\hat{Q}_j, \hat{P}_k] = i\delta_{jk} I, \qquad (j, k = 1, \ldots, n). \tag{3}$$

The shifted parity operator was discussed at length in [4, 7, 8]. Formula (2) yields the Weyl symbol of a density state. By means of an analogous formula we may define the Weyl symbol of any (bounded) operator, in general not satisfying the properties of a density state: $\rho^\dagger = \rho, \rho \geqslant 0, \operatorname{Tr} \rho = 1$. For simplicity we will call such symbols generalized Wigner functions.

The formula used by Royer makes more apparent the possibility to generalize the Wigner function as a function on the (representations of the) Weyl–Heisenberg group. This generalization allows for both the study of the transformation properties of the Wigner function under the automorphisms of the Weyl–Heisenberg group and the search for differential equations able to characterize them. To this aim, mainly to fix notations, we now briefly recall the theory of unitary irreducible representations of the Weyl–Heisenberg group.

3. Automorphisms and representations of the Weyl–Heisenberg group

Let (V, ω) be a symplectic, $2n$-dimensional real vector space and consider the $(2n + 1)$-dimensional Weyl–Heisenberg group $\mathrm{WH}(n)$ which is the central extension of the Abelian group V with respect to the two-cocycle defined by ω. The Weyl–Heisenberg group $\mathrm{WH}(n)$ can be presented as the set of pairs $g = (\mathbf{v}, t) \in V \times \mathbb{R}$ with the following composition rule:

$$g \circ g' = (\mathbf{v}, t) \circ (\mathbf{v}', t') = \left(\mathbf{v} + \mathbf{v}', t + t' + \tfrac{1}{2}\omega(\mathbf{v}, \mathbf{v}')\right). \tag{4}$$

The automorphisms group Aut(WH(n)) of the Weyl–Heisenberg group WH(n) is the set of bijections ϕ: WH(n) $\to$ WH(n) such that $\phi(g) \circ \phi(g') = \phi(g \circ g')$. The structure of the automorphisms group of the Weyl–Heisenberg group is described [9] by the following four subgroups:

(i) The automorphisms S of the linear symplectic group in $2n$ dimensions $Sp(n)$, acting on vectors of V preserving the symplectic form: $\omega(S\mathbf{v}, S\mathbf{v}') = \omega(\mathbf{v}, \mathbf{v}')$. This group has dimension $n(2n+1)$.

(ii) The inner automorphisms: $\phi_{g'}(g) = g' \circ g \circ (g')^{-1}$, that take the particular form, $\phi_{(\mathbf{v}',t')}(\mathbf{v}, t) = (\mathbf{v}, t + \omega(\mathbf{v}', \mathbf{v}))$, which is a subgroup of dimension $2n$.

(iii) The dilations $\phi_\lambda(\mathbf{v}, t) = (\lambda\mathbf{v}, \lambda^2 t)$ with one parameter.

(iv) The subgroup $\mathbb{Z}_2$ constituted of the identity and a discrete automorphism, the inversion $\imath : \imath(\mathbf{v}, t) = (A\mathbf{v}, -t)$, where A is an anti-symplectic involution on V, i.e. $\omega(A\mathbf{v}, A\mathbf{v}') = -\omega(\mathbf{v}, \mathbf{v}')$ and $A^2 = I$. Having chosen such an involution A_0, any other may be obtained as $A = SA_0S^{-1}$ by conjugating with a symplectic automorphism S. For instance, in a given Darboux chart, where $\mathbf{v} = (\mathbf{p}, \mathbf{q})$, $\mathbf{v}' = (\mathbf{p}', \mathbf{q}')$ and $\omega(\mathbf{v}, \mathbf{v}') = \mathbf{p} \cdot \mathbf{q}' - \mathbf{p}' \cdot \mathbf{q}$, A_0 may be chosen as the anti-symplectic involution $A_0(\mathbf{p}, \mathbf{q}) = (\mathbf{q}, \mathbf{p})$; also one could choose $A_0'(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, -\mathbf{q})$ and $A_0' = SA_0S^{-1}$ with $S(\mathbf{p}, \mathbf{q}) = (\mathbf{p} + \mathbf{q}, -\mathbf{p} + \mathbf{q})/\sqrt{2}$.

Any element $\phi \in$ Aut(WH(n)) can be written as a product of these four kinds of automorphisms. The group Aut(WH(n)) has dimension $(2n+1)(n+1)$.

The unitary representations of the Weyl–Heisenberg group WH(n) may be constructed from a Weyl system on the symplectic space (V, ω), that is a strongly continuous map W which associates with any vector $\mathbf{v}$ a unitary operator $W(\mathbf{v})$ acting on a Hilbert space $\mathcal{H}$ and satisfying

$$W(\mathbf{v})W(\mathbf{v}') = W(\mathbf{v} + \mathbf{v}') \exp\left(\frac{\mathrm{i}}{2}\omega(\mathbf{v}, \mathbf{v}')\right), \tag{5}$$

which implies

$$W(\mathbf{v})W(\mathbf{v}') = W(\mathbf{v}')W(\mathbf{v}) \exp(\mathrm{i}\omega(\mathbf{v}, \mathbf{v}')). \tag{6}$$

The von Neumann theorem [10] shows that it is always possible to realize the Hilbert space $\mathcal{H}$ as the space $\mathcal{L}^2(L, \mathrm{d}^n q)$ of square integrable functions with support in L, where L is any Lagrangian subspace of V with the corresponding polarization, $V = L \oplus L^*$, $(\mathbf{q}, \mathbf{p}) \in L \oplus L^*$. The unitary operators realizing the elements of the group are the usual displacement operators introduced above: $W(\mathbf{v}) = D(\mathbf{p}, \mathbf{q})$. Now, the associated canonical operators with their commutation relations $[\hat{Q}_j, \hat{P}_k] = \mathrm{i}\delta_{jk} I\,(j, k = 1, \ldots, n)$ are a realization of the Lie algebra of the generators $\{e_l\}_{l=0}^{2n}$ of the Weyl–Heisenberg group, and a unitary irreducible representation is provided by the expression:

$$U(g) = U(\mathbf{p}, \mathbf{q}, t) = D(\mathbf{p}, \mathbf{q})\,\mathrm{e}^{\mathrm{i}tI}. \tag{7}$$

In order to better put in evidence the role played by the mathematical structures involved, we prefer not to introduce the polarization $\mathbf{v} = (\mathbf{p}, \mathbf{q})$, and write the above representation in a coordinate-free form in terms of the Hermitian generators $R(e_k)$ of U, with $R(e_0) = I$, as

$$U(g) = U(\mathbf{v}, t) = W(\mathbf{v})\,\mathrm{e}^{\mathrm{i}tI} = \mathrm{e}^{\mathrm{i}R(\mathbf{v})}\,\mathrm{e}^{\mathrm{i}tI}. \tag{8}$$

The group Aut(WH(n)) acts on the space of irreducible unitary representations of WH(n) in the following way:

$$\phi^* U(g) \equiv U_\phi(g) := U(\phi(g)), \qquad \phi \in \text{Aut(WH}(n)) \tag{9}$$

and U_ϕ is a new representation.

The irreducible representations of WH(n) are parametrized up to a unitary equivalence by a real parameter γ. Kirillov's theory of coadjoint orbits [11] provides a natural way to construct them. In fact, Kirillov's theorem establishes that for nilpotent groups there is a one-to-one correspondence between coadjoint orbits of the group and equivalence classes of unitary irreducible representations of it. It is easy to check that for the Weyl–Heisenberg group the space of coadjoint orbits has two strata, the regular one whose coadjoint orbits are copies of the symplectic linear space (V, ω) and are labeled by $\gamma \neq 0$, and the singular stratum, corresponding to the label $\gamma = 0$ whose coadjoint orbits are points, hence giving rise to trivial representations. The parameter γ weights the central element of the group and it can be easily read out from a given irreducible representation looking at $U_\gamma(0,t) = \mathrm{e}^{\mathrm{i}\gamma t}$ and therefore the action of Aut$(WH(n))$ on the set of irreducible representations can be analyzed.

Symplectic and inner automorphisms form a subgroup $\mathrm{Aut}_0(\mathrm{WH}) \subset \mathrm{Aut}(\mathrm{WH})$ which does not change γ. In fact, when $\phi \in \mathrm{Aut}_0(\mathrm{WH})$ we have $U_\phi(0,t) = U(\phi(0,t)) = U(0,t)$, then there exists a unitary representation V_ϕ of $\mathrm{Aut}_0(\mathrm{WH})$ relating U_ϕ and U:

$$U_{\gamma,\phi}(g) = U_\gamma(\phi(g)) = V_\phi U_\gamma(g) V_\phi^\dagger. \tag{10}$$

For inner automorphisms $\phi_{g'}$, $V_\phi = U_\gamma(g')$ while for symplectic automorphisms S, $V_\phi = V_S = \exp\left[\mathrm{i}\left(A_S^{lm} R(e_l) R(e_m)\right)\right]\exp[\mathrm{i}\gamma t]$, where A_S^{lm} is a real symmetric $2n \times 2n$-matrix depending on S as discussed in section 5, while $R(e_k), k = 1, \ldots, 2n$, and $R(e_0) = \gamma I$ are the Hermitian generators of U_γ.

In contrast, for dilations ϕ_λ such a unitary V_ϕ does not exist

$$U_\gamma(\phi(g)) \equiv U_{\gamma,\phi}(g) \neq V_\phi U(g) V_\phi^\dagger. \tag{11}$$

In fact $U_{\gamma,\phi}(g)$ is not equivalent to $U_\gamma(g)$ since $U_{\gamma,\phi}(0,t) = \mathrm{e}^{\mathrm{i}\lambda^2\gamma t}$ is a representation labeled by $\lambda^2\gamma$. Dilations do not change the sign of γ; the inversion $\imath(\mathbf{v},t) = (A\mathbf{v}, -t)$ changes the sign of γ.

4. Wigner functions on The Weyl–Heisenberg group

The group Aut(WH) acts transitively on the space $\widehat{\mathrm{WH}}$ of equivalence classes of irreducible unitary representations of WH. On the other hand, by virtue of (7), (8), the usual Wigner function (2), may be written as

$$\mathcal{W}_\rho(\mathbf{v};\gamma=1) = 2^n\,\mathrm{Tr}[\rho U(\mathbf{v},t)\mathcal{P}U^\dagger(\mathbf{v},t)] = 2^n\,\mathrm{Tr}[\rho W(2\mathbf{v})\mathcal{P}], \tag{12}$$

and this shows that it depends on a given unitary representation, that of (7). In order to introduce a generalized definition of the Wigner function for representations with $\gamma \neq 1$, we have to choose previously a representative U_γ out of any equivalence class $[U]_\gamma$. Bearing in mind the previous analysis of the action of Aut(WH), we choose these representatives as follows:

$$\begin{aligned} U_\gamma(\mathbf{v},t) &:= U_{\gamma=1}(\sqrt{\gamma}\mathbf{v}, \gamma t) = W(\sqrt{\gamma}\mathbf{v})\,\mathrm{e}^{\mathrm{i}\gamma t} && (\gamma > 0),\\ U_\gamma(\mathbf{v},t) &:= U_{\gamma=1}(\sqrt{|\gamma|}A\mathbf{v}, -|\gamma| t) = W(\sqrt{|\gamma|}A\mathbf{v})\,\mathrm{e}^{\mathrm{i}\gamma t} && (\gamma < 0). \end{aligned} \tag{13}$$

In this way all representations are given by operators defined on the same Hilbert space, i.e., are unitary with respect to the same Hermitian structure. Since representations with negative labels γ are obtained by acting with the inversion operator on those with positive labels, in what follows we limit the discussion to the case $\gamma > 0$.

Once a representation U_γ has been chosen, the parity operator $\mathcal{P}$ may be expressed as

$$\mathcal{P} = \frac{\gamma^n}{2^n}\int \frac{\mathrm{d}^{2n}v}{(2\pi)^n} W(\sqrt{\gamma}\mathbf{v}) = \frac{1}{2^n}\int \frac{\mathrm{d}^{2n}v}{(2\pi)^n} W(\mathbf{v}). \tag{14}$$

From this expression the properties

$$\mathcal{P} = \mathcal{P}^\dagger, \qquad \mathcal{P}U_\gamma(\mathbf{v}, t)\mathcal{P} = U_\gamma(-\mathbf{v}, t) \tag{15}$$

readily follow by using (5) and (6) as well as the Dirac delta function representation:

$$\int \frac{\mathrm{d}^{2n}v}{(2\pi)^n} \exp(\mathrm{i}\omega(\mathbf{v}, \mathbf{v}')) = (2\pi)^n \delta(\mathbf{v}'). \tag{16}$$

These properties with $U_\gamma(0) = I$ yield

$$\mathcal{P}^2 = I, \qquad \mathcal{P}^{-1} = \mathcal{P} = \mathcal{P}^\dagger. \tag{17}$$

At the same time, we define the associated (generalized) Wigner function of a density state ρ as

$$\mathcal{W}_\rho(\mathbf{v}; \gamma) := 2^n \operatorname{Tr}\left[\rho U_\gamma(\mathbf{v}, t)\mathcal{P}U_\gamma^\dagger(\mathbf{v}, t)\right] = 2^n \operatorname{Tr}[\rho W(2\sqrt{\gamma}\mathbf{v})\mathcal{P}] = \mathcal{W}_\rho(\sqrt{\gamma}\mathbf{v}; 1). \tag{18}$$

We remark that, while the dependence on the parameter t disappears and the function is invariant on the subgroup $(0, t)$, a new dependence on the representation label γ appears.

As a result of this definition, the normalization property holds

$$\int \frac{\gamma^n \, \mathrm{d}^{2n}v}{(2\pi)^n} \mathcal{W}_\rho(\mathbf{v}; \gamma) = \operatorname{Tr}\left[\rho \int \frac{2^n\gamma^n \, \mathrm{d}^{2n}v}{(2\pi)^n} W(2\sqrt{\gamma}\mathbf{v})\mathcal{P}\right] = \operatorname{Tr}\rho. \tag{19}$$

In other words, the Wigner function may also be regarded as a density, i.e., the coefficient of a volume form, and in the study of the transformation properties under the group dilations one should take into account the effect of the transformation on the volume form to preserve the above normalization property.

5. Transformation properties of Wigner functions under automorphisms

We begin by considering the action of a symplectic automorphism S [12]. As we said, the action of S on a representation U_γ may be described by a unitary operator V_S, which commutes with the parity because it depends only on quadratic functions of the group generators. Then the transformed Wigner function of a state is just the Wigner function of the anti-transformed state,

$$\begin{aligned}\mathcal{W}_\rho(S\mathbf{v}; \gamma) &= 2^n \operatorname{Tr}\left[\rho V_S U_\gamma(\mathbf{v}, t) V_S^\dagger \mathcal{P} V_S U_\gamma^\dagger(\mathbf{v}, t) V_S^\dagger\right] \\ &= 2^n \operatorname{Tr}\left[V_S^\dagger \rho V_S W(2\sqrt{\gamma}\mathbf{v})\mathcal{P}\right] = \mathcal{W}_{V_S^\dagger \rho V_S}(\mathbf{v}; \gamma).\end{aligned} \tag{20}$$

In particular, we may consider the action of any generator of the symplectic group $Sp(n, \mathbb{R})$ on the symplectic space V. These actions are generated by $n(2n + 1)$ linear vector fields $X^{(\alpha)}$ which realize the Lie algebra of $Sp(n, \mathbb{R})$

$$X^{(\alpha)} = (S^{(\alpha)})_l^h x_h \frac{\partial}{\partial x_l}, \qquad \alpha = 1, \ldots, n(2n + 1), \tag{21}$$

where $(S^{(\alpha)})_l^h$ are matrices such that the products

$$(S^{(\alpha)})_l^h \omega^{lk} =: (A_{S^{(\alpha)}})^{hk} \tag{22}$$

are real symmetric matrices [13].

Integration of the vector field $\sum_\alpha \lambda_\alpha X^{(\alpha)}$ yields a one-parameter group of symplectic automorphisms:

$$\phi_{\tau,\{\lambda_\alpha\}} := \exp\left[\tau \sum_{\alpha=1}^{n(2n+1)} \lambda_\alpha X^{(\alpha)}\right]. \tag{23}$$

On the other hand, any generator is represented by a Hermitian operator $\hat{X}^{(\alpha)}$ which depends quadratically on the representation generators $R(e_k)$:

$$\hat{X}^{(\alpha)} = \sum_{h,k=1}^{2n} R(e_h)\,(A_{S^{(\alpha)}})^{hk}\,R(e_k), \tag{24}$$

where $(A_{S^{(\alpha)}})^{hk}$ is the above real symmetric $2n \times 2n$-matrix. In other words, these $n(2n+1)$ Hermitian operators are the generators of the representation V_S of the symplectic group on the Hilbert space $\mathcal{H}$. So, when $S = \phi_{\tau,\{\lambda_\alpha\}}$, the associated unitary operator is

$$V_S = \exp\left[\mathrm{i}\tau \sum_\alpha \lambda_\alpha\,(A_{S^{(\alpha)}})^{lm}\,R(e_l)R(e_m)\right]\exp[\mathrm{i}\gamma t]. \tag{25}$$

For $S_\alpha = \phi_{\tau,\alpha} := \exp[\tau X^{(\alpha)}]$, a differentiation with respect to τ at $\tau = 0$ of

$$U_{\gamma,\phi_{\tau,\alpha}}(g) = U_\gamma(\phi_{\tau,\alpha}(g)) = V_{S_\alpha} U_\gamma(g) V_{S_\alpha}^\dagger \tag{26}$$

yields

$$L_{X^{(\alpha)}} U_\gamma(g) = \mathrm{i}[\hat{X}^{(\alpha)}, U_\gamma(g)], \qquad \alpha = 1, 2, \ldots, n(2n+1). \tag{27}$$

Then, the action generated by $X^{(\alpha)}$ on the Wigner function may be written as

$$\mathcal{W}_\rho(\phi_{\tau,\alpha}(\mathbf{v}); \gamma) = \mathcal{W}_{V_{S_\alpha}^\dagger \rho V_{S_\alpha}}(\mathbf{v}; \gamma). \tag{28}$$

Differentiation with respect to τ at $\tau = 0$ gives the infinitesimal actions:

$$L_{X^{(\alpha)}} \mathcal{W}_\rho(\mathbf{v}; \gamma) - \mathcal{W}_{\mathrm{i}[\rho, \hat{X}^{(\alpha)}]}(\mathbf{v}; \gamma) = 0, \qquad \alpha = 1, \ldots, n(2n+1). \tag{29}$$

The action of an inner automorphism $\phi_{g'}$ is described by a unitary operator that is just the representation operator associated with g'. Thus

$$U_\gamma(g') U_\gamma(g) U_\gamma^\dagger(g') = U_\gamma(g)\,\mathrm{e}^{\mathrm{i}\gamma\theta} \tag{30}$$

and the Wigner function is invariant as the phase factors cancel each other:

$$\begin{aligned}\mathcal{W}_\rho(\phi_{g'}(\mathbf{v}); \gamma) &= 2^n\,\mathrm{Tr}\left[\rho U_\gamma(g') U_\gamma(\mathbf{v}, t) U_\gamma^\dagger(g') \mathcal{P} U_\gamma(g') U_\gamma^\dagger(\mathbf{v}, t) U_\gamma^\dagger(g')\right] \\ &= 2^n\,\mathrm{Tr}[\rho W(2\sqrt{\gamma}\mathbf{v})\mathcal{P}] = \mathcal{W}_\rho(\mathbf{v}; \gamma).\end{aligned} \tag{31}$$

We observe that inner automorphism actions are generated by translations on t, while the fields $X^{(\alpha)}$ on phase space vanish. So, at infinitesimal level, we get only the trivial equation:

$$\frac{\partial}{\partial t}\mathcal{W}_\rho(\mathbf{v}; \gamma) = 0. \tag{32}$$

We now consider the action of a dilation $\phi_\lambda : \phi_\lambda(\mathbf{v}, t) = (\lambda\mathbf{v}, \lambda^2 t)$. Then, as a result of our choice of the representatives U_γ, we get

$$U_\gamma(\phi_\lambda(\mathbf{v}, t)) = U_\gamma(\lambda\mathbf{v}, \lambda^2 t) = U_{\lambda^2\gamma}(\mathbf{v}, t). \tag{33}$$

So, the Wigner function transforms as

$$\mathcal{W}_\rho(\phi_\lambda(\mathbf{v}); \gamma) = \mathcal{W}_\rho(\lambda\mathbf{v}; \gamma) = 2^n\,\mathrm{Tr}[\rho U_{\lambda^2\gamma}(\mathbf{v}, t)\mathcal{P} U_{\lambda^2\gamma}(\mathbf{v}, t)] = \mathcal{W}_\rho(\mathbf{v}; \lambda^2\gamma), \tag{34}$$

while

$$\int \frac{\lambda^{2n}\gamma^n\,\mathrm{d}^{2n}v}{(2\pi)^n}\mathcal{W}_\rho(\lambda\mathbf{v}; \gamma) = \int \frac{\lambda^{2n}\gamma^n\,\mathrm{d}^{2n}v}{(2\pi)^n}\mathcal{W}_\rho(\mathbf{v}; \lambda^2\gamma) = \mathrm{Tr}\,[\rho]\,. \tag{35}$$

The dilation transformation may be more interestingly written as

$$\mathcal{W}_\rho\left(\lambda\mathbf{v}; \frac{\gamma}{\lambda^2}\right) = \mathcal{W}_\rho(\mathbf{v}; \gamma). \tag{36}$$

We observe that the dilation $(\lambda\mathbf{v}, \lambda^2 t)$ yields the expected dilation $(\lambda\mathbf{v}; \gamma/\lambda^2)$ on the label γ, which is 'dual' of the parameter t. For an infinitesimal dilation $\lambda = 1 + \epsilon$ we may expand

$$\begin{aligned}\mathcal{W}_\rho(\mathbf{v};\gamma) &= \mathcal{W}_\rho\left((1+\epsilon)\,\mathbf{v};\ \frac{\gamma}{(1+\epsilon)^2}\right)\\ &= \mathcal{W}_\rho(\mathbf{v};\gamma) + \epsilon\left[\mathbf{v}\frac{\partial\mathcal{W}_\rho}{\partial\mathbf{v}}(\mathbf{v};\gamma) - 2\gamma\frac{\partial\mathcal{W}_\rho}{\partial\gamma}(\mathbf{v};\gamma)\right] + \mathrm{O}(\epsilon^2)\end{aligned} \tag{37}$$

and obtain the following differential equation for the Wigner function:

$$\mathbf{v}\frac{\partial\mathcal{W}_\rho}{\partial\mathbf{v}}(\mathbf{v};\gamma) - 2\gamma\frac{\partial\mathcal{W}_\rho}{\partial\gamma}(\mathbf{v};\gamma) = 0. \tag{38}$$

6. Restoring Planck's constant

So far, we have put $\hbar = 1$. It is possible however to study the dependence on $\hbar$ by using the displacement operators given, instead of (3), by the expressions:

$$D(\mathbf{p},\mathbf{q}) = \exp\left[\mathrm{i}\left(\frac{\mathbf{p}}{\sqrt{\hbar}}\cdot\frac{\hat{\mathbf{Q}}}{\sqrt{\hbar}} - \frac{\mathbf{q}}{\sqrt{\hbar}}\cdot\frac{\hat{\mathbf{P}}}{\sqrt{\hbar}}\right)\right] \tag{39}$$

and the canonical commutation relations

$$\frac{1}{\hbar}[\hat{Q}_j, \hat{P}_k] = \mathrm{i}\delta_{jk}I, \qquad (j,k = 1,\ldots,n), \tag{40}$$

while t is replaced by $t/\hbar$ and the unitary representation given by (8) becomes

$$U\left(\frac{\mathbf{v}}{\sqrt{\hbar}}, \frac{t}{\hbar}\right) = W\left(\frac{\mathbf{v}}{\sqrt{\hbar}}\right)\mathrm{e}^{\mathrm{i}\frac{t}{\hbar}I} = \mathrm{e}^{\mathrm{i}\frac{R(\mathbf{v})}{\hbar}}\,\mathrm{e}^{\mathrm{i}\frac{t}{\hbar}I}, \tag{41}$$

so that eventually we get the above formulae with γ replaced by $\gamma/\hbar$ everywhere. In particular, for the Wigner function we have

$$\mathcal{W}_\rho(\sqrt{\gamma}\mathbf{v};1) = \mathcal{W}_\rho(\mathbf{v};\gamma) \longrightarrow \mathcal{W}_\rho\left(\sqrt{\frac{\gamma}{\hbar}}\mathbf{v};1\right) = \mathcal{W}_\rho\left(\mathbf{v};\frac{\gamma}{\hbar}\right). \tag{42}$$

Under the action of a dilation, $(\lambda\mathbf{v}, \lambda^2 t) \to (\lambda\mathbf{v}; \gamma/\lambda^2) \to (\lambda\mathbf{v}; \gamma/\lambda^2\hbar)$ and we may choose $\gamma = 1$, to get a differential equation for the Wigner function $\mathcal{W}_\rho\left(\mathbf{v};\frac{1}{\hbar}\right)$ corresponding to the infinitesimal 'dilation' $(\lambda\mathbf{v}; 1/\lambda^2\hbar)$. However, we should remark here that an equation such as

$$\mathbf{v}\frac{\partial\mathcal{W}_\rho}{\partial\mathbf{v}}(\mathbf{v};\gamma) + 2\hbar\frac{\partial\mathcal{W}_\rho}{\partial\hbar}(\mathbf{v};\gamma) = 0 \tag{43}$$

cannot hold in general, in contrast with (38), because the functional dependence of ρ on $\hbar$ varies from state to state. So, we get different differential equations for different states, and the above equation, for instance, holds only when the density state ρ depends on $(1/\hbar)^n$ for a system with n degrees of freedom.

To clarify this point we now discuss two different examples. We consider two physical states: the ground state of a three-dimensional isotropic harmonic oscillator and the ground state of a hydrogen atom (with dimensional units restored).

Example 1. For a harmonic oscillator of mass m and frequency ω one has a characteristic unit length $\ell = \sqrt{\hbar/m\omega}$ and the ground state wavefunction is

$$\varphi_0(\mathbf{r}) = \pi^{-\frac{3}{4}}\ell^{-\frac{3}{2}}\exp\left(-\frac{r^2}{2\ell^2}\right). \tag{44}$$

The corresponding Wigner function reads

$$\mathcal{W}_{\varphi_0}(\mathbf{p},\mathbf{q};\hbar) = 8\exp\left(-\frac{q^2}{\ell^2} - \frac{\ell^2 p^2}{\hbar^2}\right) = 8\exp\left(-\frac{m\omega}{\hbar}q^2 - \frac{(m\omega)^{-1}}{\hbar}p^2\right), \tag{45}$$

so one can see at once that this function satisfies the partial differential equation,

$$\mathbf{p}\frac{\partial \mathcal{W}_{\varphi_0}}{\partial \mathbf{p}} + \mathbf{q}\frac{\partial \mathcal{W}_{\varphi_0}}{\partial \mathbf{q}} + 2\hbar\frac{\partial \mathcal{W}_{\varphi_0}}{\partial \hbar} = 0, \tag{46}$$

which reflects the scaling properties associated with the transformation properties of the Wigner function under a dilation, when $\gamma \to 1/\hbar$. The above differential equation is satisfied by all Wigner functions of the excited states of the harmonic oscillator, as well as by those of pure states expressible as their superposition with coefficients independent of $\hbar$ as, e.g., the Wigner function of coherent states or of convex sums of the harmonic oscillator eigenstates.

Example 2. In the case of the hydrogen atom, the ground-state normalized wavefunction is

$$\psi_0(\mathbf{r}) = \pi^{-\frac{1}{2}} a_B^{-\frac{3}{2}} \exp\left(-\frac{r}{a_B}\right), \tag{47}$$

where a_B is the Bohr radius

$$a_B = \frac{\hbar^2}{me^2}, \tag{48}$$

with m, e being the mass and electric charge of the electron, respectively. The corresponding Wigner function reads

$$\mathcal{W}_{\psi_0}(\mathbf{p},\mathbf{q};\hbar) = \frac{1}{\pi a_B^3}\int_{\mathbb{R}^3} \mathrm{d}^3x \exp\left(-\mathrm{i}\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar}\right)\exp\left[-\frac{\left|\mathbf{q}+\frac{\mathbf{x}}{2}\right|}{a_B} - \frac{\left|\mathbf{q}-\frac{\mathbf{x}}{2}\right|}{a_B}\right], \tag{49}$$

so it depends on the ratio $\sim q/\hbar^2$ and the product $\sim p\hbar$. As a consequence, the Wigner function is invariant under the scaling $(\mathbf{p},\mathbf{q},\hbar) \to (\lambda^{-1}\mathbf{p}, \lambda^2\mathbf{q}, \lambda\hbar)$:

$$\mathcal{W}_{\psi_0}(\lambda^{-1}\mathbf{p}, \lambda^2\mathbf{q}, \lambda\hbar) = \mathcal{W}_{\psi_0}(\mathbf{p},\mathbf{q};\hbar). \tag{50}$$

At the infinitesimal level, this yields the differential equation

$$-\mathbf{p}\frac{\partial \mathcal{W}_{\psi_0}}{\partial \mathbf{p}} + 2\mathbf{q}\frac{\partial \mathcal{W}_{\psi_0}}{\partial \mathbf{q}} + \hbar\frac{\partial \mathcal{W}_{\psi_0}}{\partial \hbar} = 0, \tag{51}$$

which is different from that satisfied by the harmonic oscillator Wigner function (46). Of course, the above equation is satisfied also by the Wigner functions of the excited states of the hydrogen atom as well as of their convex superpositions, with coefficients independent of $\hbar$.

7. Tomograms and Weyl–Heisenberg group automorphisms

In this section, we study the action on symplectic tomograms of the group of automorphisms of the Weyl–Heisenberg group $\mathrm{Aut}(\mathrm{WH}(n))$.

Again we put $\hbar = 1$. We will just consider $n = 1$, as the general case follows straightforwardly. We recall that, given a density state ρ, its symplectic tomogram $\widetilde{\mathcal{W}}_\rho(X,\mu,\nu)$ depends on the real parameters X, μ, ν and is defined by

$$\widetilde{\mathcal{W}}_\rho(X,\mu,\nu) = \mathrm{Tr}[\rho\delta(X - \mu\hat{Q} - \nu\hat{P})]. \tag{52}$$

The symplectic tomogram above can be written as the Radon transform of the Wigner function $\mathcal{W}_\rho(p,q;\gamma=1)$ of the density matrix ρ:

$$\widetilde{\mathcal{W}}_\rho(X,\mu,\nu) = \int_{\mathbb{R}^2} \mathcal{W}_\rho(p,q;\gamma=1)\delta(X-\mu q-\nu p)\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi}. \tag{53}$$

The normalization of the Wigner function of the density state yields the normalization of the tomogram:

$$\int \widetilde{\mathcal{W}}_\rho(X,\mu,\nu)\,\mathrm{d}X = 1. \tag{54}$$

The density state ρ is expressed in terms of its tomogram as

$$\rho = \frac{1}{2\pi}\int_{\mathbb{R}^3} \widetilde{\mathcal{W}}_\rho(X,\mu,\nu)\exp[\mathrm{i}(X-\mu\hat{Q}-\nu\hat{P})]\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu, \tag{55}$$

and this corresponds to writing the Wigner function as the Radon anti-transform of the tomogram

$$\mathcal{W}_\rho(p,q;\gamma=1) = \frac{1}{2\pi}\int_{\mathbb{R}^3} \widetilde{\mathcal{W}}_\rho(X,\mu,\nu)\exp[\mathrm{i}(X-\mu q-\nu p)]\,\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu. \tag{56}$$

By means of the above relations we are now able to study the action of the automorphisms of the Weyl–Heisenberg group onto symplectic tomograms. In fact, we may use the Radon transform to define

$$\phi^*\widetilde{\mathcal{W}}_\rho(X,\mu,\nu) = \widetilde{\mathcal{W}}_\rho(\phi(X,\mu,\nu)) := \int_{\mathbb{R}^2} \mathcal{W}_\rho(\phi(p,q;\gamma=1))\delta(X-\mu q-\nu p)\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi} \tag{57}$$

and then use the transformation properties of the Wigner functions discussed in section 5 to obtain the corresponding tomographic properties.

For instance, in the case of the action of the symplectic automorphism $S \in Sp(1,\mathbb{R})$ on the phase space V, $\phi(p,q) = S\mathbf{v}$, one obtains

$$\begin{aligned}\widetilde{\mathcal{W}}_\rho(S(X,\boldsymbol{\mu})) &= \int \mathcal{W}_\rho(S\mathbf{v})\delta(X-\boldsymbol{\mu}\cdot\mathbf{v})\frac{\mathrm{d}^2v}{2\pi} = \int \mathcal{W}_\rho(\mathbf{v})\delta(X-\boldsymbol{\mu}\cdot S^{-1}\mathbf{v})\frac{\mathrm{d}^2v}{2\pi}\\ &= \int \mathcal{W}_\rho(\mathbf{v})\delta(X-((S^{-1})^{\mathrm{T}}\boldsymbol{\mu})\cdot\mathbf{v})\frac{\mathrm{d}^2v}{2\pi} = \widetilde{\mathcal{W}}_\rho(X,(S^{-1})^{\mathrm{T}}\boldsymbol{\mu}),\end{aligned} \tag{58}$$

where $\boldsymbol{\mu} = (\nu,\mu)$. On the other hand, by Radon transforming the right-hand side of (20), we get

$$\widetilde{\mathcal{W}}_\rho(X,(S^{-1})^{\mathrm{T}}\boldsymbol{\mu}) = \widetilde{\mathcal{W}}_\rho(S(X,\boldsymbol{\mu})) = \widetilde{\mathcal{W}}_{V_S^\dagger\rho V_S}(X,\boldsymbol{\mu}). \tag{59}$$

Analogously, under inner automorphisms, the tomogram $\widetilde{\mathcal{W}}_\rho$ is invariant, just as the Wigner function.

Finally, we may generalize the definition of symplectic tomogram by using the Wigner function $\mathcal{W}_\rho(\mathbf{v};\gamma)$ in the Radon transform. This introduces a dependence on the representation label γ:

$$\widetilde{\mathcal{W}}_\rho(X,\boldsymbol{\mu};\gamma) = \widetilde{\mathcal{W}}_\rho\left(X,\frac{1}{\sqrt{\gamma}}\boldsymbol{\mu};1\right), \tag{60}$$

which stems out from

$$\begin{aligned}\widetilde{\mathcal{W}}_\rho(X,\boldsymbol{\mu};\gamma) &:= \int \frac{\sqrt{\gamma}\,\mathrm{d}^2v}{2\pi}\mathcal{W}_\rho(\mathbf{v};\gamma)\delta(X-\boldsymbol{\mu}\cdot\mathbf{v})\\ &= \int \frac{\mathrm{d}^2v}{2\pi}\mathcal{W}_\rho(\mathbf{v};\gamma=1)\delta\left(X-\frac{\boldsymbol{\mu}}{\sqrt{\gamma}}\cdot\mathbf{v}\right).\end{aligned} \tag{61}$$

In this way, any scaling $(\mathbf{v};\gamma)\mapsto(\lambda\mathbf{v};\gamma)$, that induces the transformation $\mathcal{W}_\rho(\lambda\mathbf{v};\gamma)\to\mathcal{W}_\rho(\mathbf{v};\lambda^2\gamma)$, will give

$$\widetilde{\mathcal{W}}_\rho\left(X,\frac{\boldsymbol{\mu}}{\lambda};\gamma\right) = \widetilde{\mathcal{W}}_\rho(X,\boldsymbol{\mu};\lambda^2\gamma), \tag{62}$$

which leads to

$$\widetilde{\mathcal{W}}_\rho\left(X, \frac{\boldsymbol{\mu}}{\lambda}; \frac{\gamma}{\lambda^2}\right) = \widetilde{\mathcal{W}}_\rho(X, \boldsymbol{\mu}; \gamma). \tag{63}$$

Of course, as was discussed in the previous sections, also the above tomographic transformation properties have associated differential equations.

We conclude this section by remarking that in the case of the multi-mode system, one could also use different parameters γ_k for each kth mode contribution to the Wigner function. For instance, this possibility was used to formulate a separability criterion for Gaussian photon states [14], but in this paper we consider only homogeneous scaling law with all γ_k's equal to the same γ, that is

$$\widetilde{\mathcal{W}}_\rho(\mathbf{X}, \boldsymbol{\mu}; \gamma) = \int \frac{\gamma^n \,\mathrm{d}^n p \,\mathrm{d}^n q}{(2\pi)^n} \mathcal{W}_\rho(\mathbf{p}, \mathbf{q}; \gamma) \prod_{k=1}^{n} \delta(X_k - \mu_k q_k - \nu_k p_k). \tag{64}$$

8. Concluding remarks

Wigner functions play a prominent role in the formulation of quantum mechanics on phase space, Wigner considered them associated with states or wavefunctions. Their generalization from pure states to mixed states and further to any suitable operator, provides a very interesting setting to deal with noncommutative geometry [15–17]. In particular, the Wigner function turns out to be very useful in the study of the quantum-classical transition [18]. From all these applications it is therefore not surprising that many generalizations have been proposed and elaborated in the existing literature.

As usual, generalizations identify some aspects which seem to be desirable and useful and try to propose an enlarged setting where these aspects are preserved. Our formulae (2) and (12) allow us to discuss more clearly the relation of our generalization to some others we are aware of.

Our idea of using the parity operator to deal with the Wigner function, motivated from Royer [7] and Grossmann [19] (but see also [20–22]), uses the fact that the parity operator, representing reflections about the origin of the symplectic vector space (the Abelian vector group we are considering), may be translated with the help of the displacement operator to provide reflections about any point of the 'affine space'. The family of all these operators, one for each point, constitutes what we have called elsewhere a tomographic set [23, 24], this set separates states. The Wigner function is given by the expectation value of these reflection operators on the state we are dealing with, this association depends on the point and on the representation we are using.

If the similarity transformation acting on the parity operator is transferred to the state, one gets an orbit in the coadjoint representation. This aspect is what Ali and collaborators [25–27] have considered most fundamental to present their generalization. They rely on the results by Kirillov [11] relating coadjoint orbits and unitary irreducible representations. To go from one orbit to another, or from one representation to another, they introduce some appropriate conditions on the group and the orbits. For a recent paper dealing with some of these aspects in the framework of wavelets and phase space see [28].

The generalization proposed by Tate [29, 30], but see also Mukunda and collaborators [31, 32], uses the fact that the point about which the previous reflections are performed may be identified as the 'mid-point' of a geodesic on a symmetric space, this remark allows us to implement an operator on the setting of symmetric spaces with properties similar to those of the displaced parity operator used in this paper.

The merit of our approach is to show very clearly that the Wigner function is defined on a group and on the space of its unitary irreducible representations, therefore we may use this as a starting point to generalize the construction also to other finite dimensional groups along the lines of [33]. As is shown in that paper, this kind of generalization may be quite useful to apply generalized Wigner functions to deal with quantum computation and quantum information [34].

Other possible generalizations, on which we are presently working, consider unitary operators associated with other finite-dimensional subgroups (replacing the parity operator by an appropriate choice) with the requirement that the automorphism group of the space on which we want to define the generalized Wigner function, acts on this subgroup in a way that gives rise to a tomographic set of operators. We shall come back to these points in a forthcoming paper.

Acknowledgments

A Ibort acknowledges the support from SIMUMAT project and MTM2007-62478. V I Man'ko thanks INFN and University 'Federico II' of Naples for their hospitality and Russian Foundation for basic research for partial support under project Nos. 07-02-00598 and 08-02-90300-Viet.

References

[1] Wigner E 1932 *Phys. Rev.* **40** 749–59
[2] Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover)
[3] von Neumann J 1927 *Nachr. Ges. Wiss. Göttingen* **11** 245
[4] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882–902
[5] Radon J 1917 *Ber. Verh. Sachs. Akad.* **69** 262
[6] Mancini S, Man'ko V I and Tombesi P 1996 *Phys. Lett.* A **213** 1–6
[7] Royer A 1977 *Phys. Rev.* A **15** 449–50
[8] Vourdas A 2006 *J. Phys. A: Math. Gen.* **39** R65–R141
[9] Folland G B 1989 *Harmonic Analysis in Phase Space* (Princeton, NJ: Princeton University Press)
[10] Esposito G, Marmo G and Sudarshan E C G 2004 *From Classical to Quantum Mechanics* (Cambridge: Cambridge University Press)
[11] Kirillov A A 2004 *Lectures on the Orbit Method (Graduate Studies in Mathematics)* vol 64 (Providence, RI: American Mathematical Society)
[12] Ercolessi E, Marmo G, Morandi G and Mukunda N 2007 *J. Phys.: Conf. Ser.* **87** 012010
[13] Marmo G, Morandi G, Simoni A and Ventriglia F 2002 *J. Phys. A: Math. Gen.* **35** 8393–406
[14] Man'ko O V, Man'ko V I, Marmo G, Sudarshan E C G and Zaccaria F 2006 *Phys. Lett.* A **357** 255–60
[15] Connes A 1994 *Noncommutative Geometry* (San Diego, CA: Academic)
[16] Landi G 1997 *An Introduction to Noncommutative Spaces and Their Geometries (Lecture Notes in Physics, New Series m: Monographs,* vol 51*)* (Berlin: Springer)
[17] Gracia-Bondia J M, Lizzi F, Marmo G and Vitale P 2002 *J. High Energy Phys.* JHEP04(2002)026
[18] Marmo G, Scolarici G, Simoni A and Ventriglia F 2005 *Int. J. Geom. Methods Mod. Phys.* **2** 127–45
[19] Grossmann A 1976 *Commun. Math. Phys.* **48** 191–4
[20] Mukunda N, Arvind, Chaturvedi S and Simon R 2004 *J. Math. Phys.* **45** 114–48
[21] Mukunda N, Marmo G, Zampini A, Chaturvedi S and Simon R 2005 *J. Math. Phys.* **46** 012106
[22] Chaturvedi S, Ercolessi E, Marmo G, Morandi G, Mukunda N and Simon R 2005 Phase-space descriptions of operators and the Wigner distribution in quantum mechanics: I. A Dirac inspired view arXiv:quant-ph/0507053v2
[23] Man'ko V I, Marmo G, Simoni A, Stern A, Sudarshan E C G and Ventriglia F 2006 *Phys. Lett.* A **351** 1–12
[24] Man'ko V I, Marmo G, Simoni A and Ventriglia F 2006 *Open Syst. Inf. Dyn.* **13** 239–53
[25] Ali S T, Atakishiyev N M, Chumakov S M and Wolf K B 2000 *Ann. Inst. Henri Poincaré* **1** 685–714
[26] Krasowska A E and Ali S T 2003 *J. Phys. A: Math. Gen.* **36** 2801–20
[27] Ali S T, Führ H and Krasowska A E 2003 *Ann. Inst. Henri Poincaré* **4** 1015–50

[28] Aniello P, Man'ko V I and Marmo G 2008 *J. Phys. A: Math. Theor.* **41** 285304

[29] Tate T 2001 *Noncommutative Differential Geometry and its Applications to Physics, Proc. Workshop held in Shonan, 31 May–4 June 1999 (Mathematical Physics Studies,* vol 23*)* ed Y Maeda *et al* (Dordrecht: Kluwer) pp 227–43

[30] Tate T 2002 *Ann. Global Anal. Geom.* **22** 29–48

[31] Chaturvedi S, Ercolessi E, Marmo G, Morandi G, Mukunda N and Simon R 2006 *J. Phys. A: Math. Gen.* **39** 1405–23

[32] Chaturvedi S, Marmo G, Mukunda N, Simon R and Zampini A 2006 *Rev. Math. Phys.* **18** 887–912

[33] Chaturvedi S, Ercolessi E, Marmo G, Morandi G, Mukunda N and Simon R 2005 *Pramana* **65** 981–93

[34] Narnhofer H 2006 *J. Phys. A: Math. Gen.* **39** 7051–64